

# Hypergeometric tail inequalities: ending the insanity

Matthew Skala  
[mskala@ansuz.sooke.bc.ca](mailto:mskala@ansuz.sooke.bc.ca)

March 22, 2009

## 1 Introduction

I recently needed to put a tail inequality on an hypergeometric distribution. This should be an easy thing to do; but I found the available online sources to be really frustrating. Everybody uses different notation, and most people seem to like giving helpful examples in which the word “success” is used to describe failure and vice versa, and the whole thing is likely to drive the reader nuts. Here, for my own future reference and for the benefit of anyone trying to do the same thing, is a summary of what I was able to glean in what I hope will be clearly understandable terms.

I don’t claim there’s any original math in these notes; this is just a summary of well-known results; but it cost me a fair bit of annoyance to get issues like notation straightened out. If you use these notes, a citation would be appreciated. See this document’s bibliography for the appropriate citation form [3].

It is assumed that you know about as much as I did about this stuff before I did the research: namely, you should know enough to know that applying a tail inequality to a hypergeometric distribution is what you want to do, even if you have trouble keeping track of the parameters of the distribution or knowing exactly which tail inequality you want. You’re also expected to be mentally flexible enough to translate the balls-and-urn description into whatever your real application is. I’ll spare you the confusing burned-out-lightbulbs example. My own actual application had to do with counting bits in the bitwise AND and OR of random bit strings with known numbers of 1 bits.

The articles by Chvátal and Hoeffding may be hard to find online, especially if you don’t have academic library privileges [1, 2]. Contact me by email if you need help locating them.

## 2 Setup and notation

You’ve got an urn with  $N$  balls in it. Some of them, namely  $M$  of them, are white. The rest, namely  $N - M$  of the balls, are black. You’re going to draw

out  $n$  balls from the urn. You are drawing them *uniformly*, which means that every time you pull out a ball it is equally likely to be any of the balls in the urn at that moment. However, you are drawing them *without replacement*, which means that after you've drawn out a ball of one colour, you've reduced the number of balls of that colour remaining and so the next one will be a little more likely to be the other colour. If instead you threw each ball back in after drawing it, then every draw would have the same chances, we'd be dealing with the geometric distribution instead of the hypergeometric distribution, and the math would be a lot easier. But this time you're drawing without replacement.

Now, how many white balls are you going to get among the  $n$  you draw? Let's call this number  $i$ ; the question is what interesting things we can say about the distribution of the random variable  $i$ , which is called an hypergeometric distribution.

The short answer is that you will get about the same fraction of white balls in your  $n$ -ball sample as the fraction of white balls among the  $N$  that the urn contained at the start. That's the expected value of  $i$ . Moreover, you will nearly always get very close to exactly that fraction. The distribution has light tails. It isn't a normal distribution bell curve (which is approached by a geometric distribution, which in turn is what you'd get by sampling with replacement) but it does have the same kind of faster-than-exponential fall-off that you would get from the normal distribution. As a result you can put a limit just a little bit above the expected value of  $i$  and say " $i$  is nearly always below this limit" or put another limit just a little below and say " $i$  is nearly always above this limit." That is what a tail inequality does.

The usual suspects (MathWorld [4] and Wikipedia [5]) and their sources use many different notations. I am following the notation for variables used by Chvátal [1], because his paper seemed easiest to understand. If you try to read the encyclopedia entries, you can try to translate using this table:

	Chvátal [1]	MathWorld [4]	Wikipedia [5]
	and these notes		
balls in urn	$N$	$n + m$	$N$
balls that count	$M$	$n$	$m$
balls that don't count	$N - M$	$m$	$N - m$
balls you draw	$n$	$N$	$n$
drawn balls that count	$i$	$i$	$k$

### 3 The distribution

What's the chance of getting exactly  $i$  white balls? For that we want the probability distribution function; Chvátal doesn't give a notation for it but I am using one based on his notation for the cumulative distribution function:

$$h(M, N, n, i) = \binom{M}{i} \binom{N - M}{n - i} / \binom{N}{n} \quad (1)$$

That follows from simple counting: how many ways can we draw out  $n$  balls including exactly  $i$  of the  $M$  white balls, compared to the number of ways we

can draw out  $n$  balls without caring about how many of them are white? The answer is that we must choose  $i$  of the  $M$  white balls to draw, hence the factor of  $\binom{M}{i}$ , and  $n - i$  of the  $N - M$  black balls, hence  $\binom{N-M}{n-i}$ , and then divide that by  $\binom{N}{n}$  for drawing  $n$  of the  $N$  balls without regard to colour. (All these choices are uniform.)

The expected value is just the same fraction of white balls in the sample as in the urn:

$$E[i] = n \frac{M}{N} \quad (2)$$

and the variance is as follows:

$$V[i] = n \frac{M(N-M)(N-n)}{N^2(N-1)} \quad (3)$$

Proofs for mean and variance are in MathWorld [4].

## 4 Useful symmetries

The Wikipedia article (as of this writing, of course; Wikipedia is a moving target) gives some useful symmetries [5]. In our notation:

$$h(M, N, n, i) = h(N - M, M, n, n - i) \quad (4)$$

$$h(M, N, n, i) = h(M, N, N - n, M - i) \quad (5)$$

$$h(M, N, n, i) = h(n, N, M, i) \quad (6)$$

If you have  $M$  balls white, draw  $n$ , and hope for  $i$  of them to be white, you could instead flip all the colours, draw  $n$ , and hope for  $n - i$  of them to be white (4). Also, if you draw  $n$  balls and find  $i$  to be white, that's the same as finding the  $M - i$  remaining white balls among the  $N - n$  you did not draw; you can swap “drawn” and “not drawn” balls (5). Finally, you can swap the concepts of “drawn” and “coloured white” and imagine that the urn is choosing  $M$  balls to possibly be drawn by you, instead of you choosing  $n$  balls to possibly be coloured white in the urn (6).

## 5 Tail inequalities

We're interested in the chance that  $i$  is at least  $k$ , for some  $k$  that will be a little bigger than the expected value  $E[i]$ . We want to say that when  $k$  is just a tiny bit bigger than  $E[i]$ , then this chance is already very small. That will mean proving that this function is small:

$$H(M, N, n, k) = \sum_{i=k}^n h(M, N, n, i) = \sum_{i=k}^n \binom{M}{i} \binom{N-M}{n-i} / \binom{N}{n} \quad (7)$$

That's the sum for all  $i \geq k$  of the probability distribution function  $h(M, N, n, i)$ ; we could equally correctly write the summation as going to infinity, because  $h(M, N, n, i)$  is zero for  $i > n$ ; I wrote it up to  $n$  for consistency with Chvátal [1].

Chvátal gives the following bound, which he credits to Hoeffding [1, 2]. I believe this is a special case of the well-known result now known as Hoeffding’s Inequality, but that’s a very powerful result and the steps required to apply it to the hypergeometric distribution in particular are a little involved. Where  $p = M/N$  and  $k = (p + t)n$  with  $t \geq 0$ , we have this:

$$H(M, N, n, k) \leq \left( \left( \frac{p}{p+t} \right)^{p+t} \left( \frac{1-p}{1-p-t} \right)^{1-p-t} \right)^n \quad (8)$$

That is a bit of a mess, but we can relax it a little further to get what Chvátal describes as a “more elegant but weaker” bound which is more likely what we’ll want to use when applying this result: [1]

$$H(M, N, n, k) \leq e^{-2t^2n} \quad (9)$$

That’s a nice one-sided tail inequality for hypergeometric distributions. Stating it in terms that sound like what we want for using it in proving that a randomized algorithm works: If  $i$  is a hypergeometric random variable with the parameters  $N$ ,  $M$ , and  $n$  as described above, then

$$Pr[i \geq E[i] + tn] \leq e^{-2t^2n} \quad (10)$$

If we want an inequality for the other tail, then we can apply the symmetry (4) as follows:

$$Pr[i \leq k'] = \sum_{i=0}^{k'} h(M, N, n, i) \quad (11)$$

$$= \sum_{i=0}^{k'} h(N - M, N, n, n - i) \quad (12)$$

Then we can change the index of summation to  $j = n - i$  and get:

$$Pr[i \leq k'] = \sum_{j=n-k'}^n h(N - M, N, n, j) \quad (13)$$

The other side’s inequality (9) can give us a nice bound for that if we choose  $k$ ,  $t$ , and  $p$  properly. We want  $k = n - k' = (p + t)n$  where  $p = (N - M)/N = 1 - (M/N)$ . Then doing the algebra we get  $k' = E[i] - tn$ , nicely equal and opposite to the other side’s bound:

$$Pr[i \leq E[i] - tn] \leq e^{-2t^2n} \quad (14)$$

## References

- [1] V. Chvátal. The tail of the hypergeometric distribution. *Discrete Mathematics*, 25(3):285–287, 1979.

- [2] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.
- [3] Matthew Skala. Hypergeometric tail inequalities: ending the insanity, 2009. Online <http://ansuz.sooke.bc.ca/professional/hypergeometric.pdf>.
- [4] Eric W. Weisstein. Hypergeometric distribution. From MathWorld—A Wolfram Web Resource. Online <http://mathworld.wolfram.com/HypergeometricDistribution.html>.
- [5] Wikipedia. Hypergeometric distribution. Revision 273333657, 26 February 2009. Online [http://en.wikipedia.org/w/index.php?title=Hypergeometric\\_distribution&oldid=273333657](http://en.wikipedia.org/w/index.php?title=Hypergeometric_distribution&oldid=273333657).